Chip-Firing on Signed Graphs

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Abstract

Chip-firing is a simple game played on a graph G, where chips are placed on the vertices of G and distributed according to a simple rule. Certain stable configurations of chips define the *critical group* of G, and the dynamics of chip-firing has found applications in mathematics, physics, and economics. We study chip-firing and critical groups for *signed graphs*, modeling a scenario that involves both cooperative and antagonistic interactions. For this we apply the Guzmán-Klivans theory of chip-firing on general invertible matrices.

Signed Graphs and Their Laplacians

Definition 1. A *signed graph* G_ϕ consists of a graph $G = (V, E)$ equipped with a signature $\phi : E \to \{+, -\}$. We let $|G|$ denote underlying graph with all positive edges.

Definition 2. Let $G = G_\phi$ be a signed graph with sink vertex q and non-sink vertices $\{v_1, \dots, v_n\}$. The *Laplacian* $L = L_G$ is the $n \times n$ matrix given by $D - A$, where D is the diagonal matrix with entries $d_{ii} = \deg(v_i)$, and A is the adjacency matrix of G

$$a_{ij} = \begin{cases} 1 & \phi(\{v_i, v_j\}) = + \\ -1 & \phi(\{v_i, v_j\}) = - \end{cases}.$$

We let M denote the Laplacian of $|G|$.

The Critical Group

For a signed graph $G = G_\phi$, the *critical group* is defined as

$$\mathcal{K}(G) = \mathbb{Z}^n / \mathrm{im}(L_G),$$

where L_G is the signed Laplacian.

The order $|\mathcal{K}(G)|$ of the critical group is related to the number of generalized spanning trees of G (more precisely the number of bases of its underlying *matroid*).

Valid Configurations and Chip-Firing

Definition 3. A *configuration* on a signed graph G_ϕ is a vector $\vec{c} = (c_1\ c_2\ \dots\ c_n) \in \mathbb{Z}^n_{\geq 0}$ encoding the numbers of chips on each non-sink vertex $v_1, v_2, \dots, v_n$.

We employ the Guzmán-Klivans theory of 'chip-firing on invertible matrices' [1]. The Laplacians $L = L_{G_\phi}$ and $M = L_{|G|}$ define the cone of *valid configurations*.

$$S^+ = \{LM^{-1}\vec{x} : \vec{x} \geq \vec{0}\} \cap \mathbb{Z}^n.$$

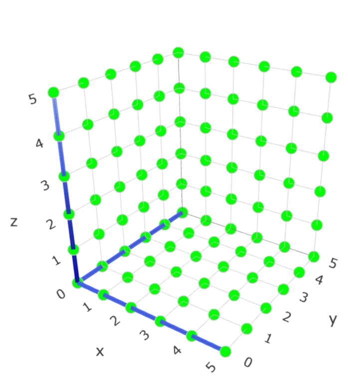


Figure 1: Valid Configurations of $|G|$

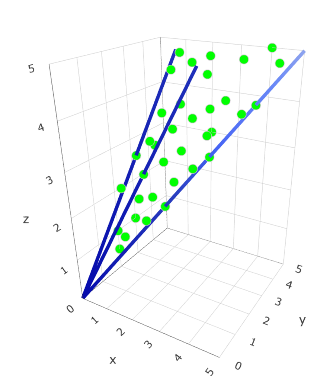


Figure 2: Valid Configurations of G_ϕ

Definition 4. Suppose $G = G_\phi$ is a signed graph and $\vec{c}$ is a valid configuration. A vertex $v_i \in V(G)$ can *fire* if

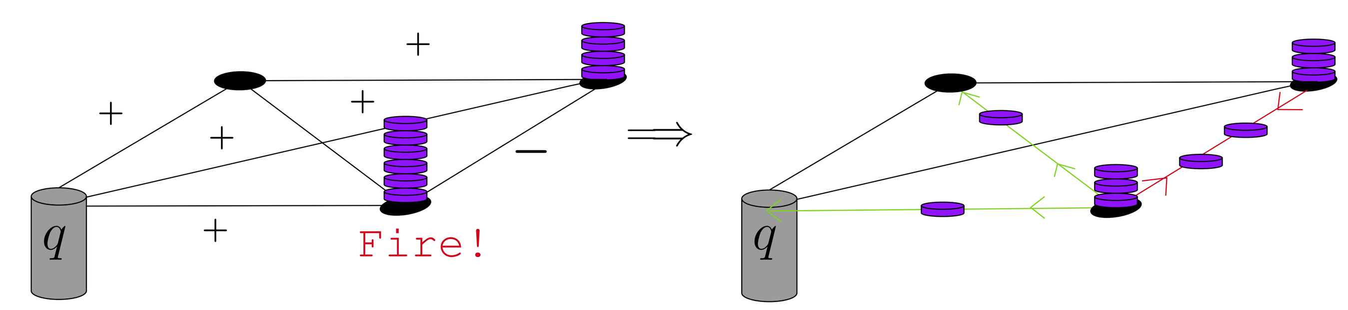
$$\vec{d} = \vec{c} - L_G e_i$$

is valid. Similarly, a multiset of vertices $\{v_{i_1}, \dots, v_{i_k}\}$ can *multi-fire* if

$$\vec{d} = \vec{d} - L_G \left(\sum_{j=1}^{k} e_{i_j} \right)$$

is valid.

Chip-Firing Example



$$\underbrace{\begin{pmatrix} 0 \\ 4 \\ 6 \end{pmatrix}}_{\vec{c}} - \underbrace{\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & +1 \\ -1 & +1 & 3 \end{pmatrix}}_{L_G} \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{e_3} = \underbrace{\begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}}_{\vec{c}'} \in \mathrm{im}\,(L_G)$$

Special Configurations

Definition 5. A valid configuration $\vec{c}$ is *critical* if

- $\vec{c}$ is stable (no vertex can fire)
- there exists a $\vec{b}$ where each vertex can fire, and which stabilizes to $\vec{c}$.

Addition of critical configurations recovers the critical group $\mathcal{K}(G)$ of the signed graph $G = G_\phi$. We let $I_{\mathcal{K}(G)}$ denote the identity element.

Definition 6. A valid configuration is *z-superstable* if one cannot legally multi-fire a cluster of sites, $v_{i_1}, \dots, v_{i_k}$.

From [1] we know that each equivalence class in $\mathbb{Z}^n/\mathrm{im}(L)$ has exactly one critical and one z-superstable configuration.

Case Study: Cycles and Wheels

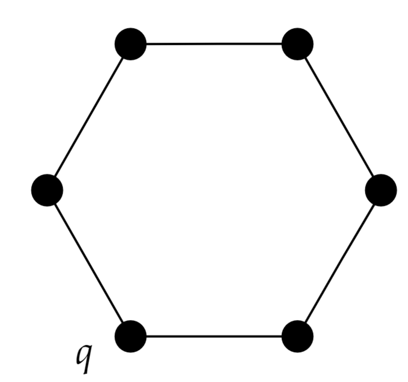


Figure 3: The Cycle Graph C_6

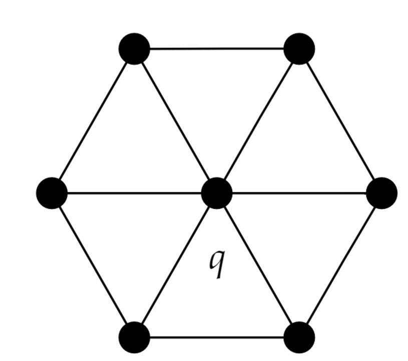


Figure 4: The Wheel Graph W_6

Critical Groups of Cycles and Wheels

Theorem 1. *For any signed cycle $(C_n)_\phi$ we have $\mathcal{K}((C_n)_\phi) \cong \mathbb{Z}_n$.*

Theorem 2. ***[Duality between z-superstables and criticals]*** *Let $-C_{2n+1}$ denote the odd cycle with only negative edges. For any n the set map*

$$f : \{z\text{-superstable configurations}\} \to \{\text{critical configurations}\}$$

given by $\vec{c} \mapsto I_{\mathcal{K}(-C_{2n+1})} + \vec{c}$ is a bijection.

Theorem 3. *Let $n \geq 3$. For any singed wheel $W_n = (W_n)_\phi$, we have:*

$$\mathcal{K}(W_n) = \begin{cases} \mathbb{Z}_{f_n} \oplus \mathbb{Z}_{5f_n} & n \text{ is odd and } W_n \text{ is unbalanced} \\ \mathbb{Z}_{f_n} \oplus \mathbb{Z}_{5f_n} & n \text{ is even and } W_n \text{ is balanced} \\ \mathbb{Z}_{\ell_n} \oplus \mathbb{Z}_{\ell_n} & n \text{ is odd and } W_n \text{ is balanced} \\ \mathbb{Z}_{\ell_n} \oplus \mathbb{Z}_{\ell_n} & n \text{ is even and } W_n \text{ is unbalanced} \end{cases}$$

where f_n denotes the nth Fibonacci number and ℓ_n the nth Lucas number.

Stability Results

Theorem 4. ***[Checking for superstability and criticality]*** *Suppose (L, M) is a integral chip-firing pair, where L is an invertible matrix and M is an M-matrix. Suppose $\vec{c} \in S^+$ is a valid configuration. Then $\vec{c}$ is z-superstable if and only if $\lfloor ML^{-1}\vec{c} \rceil$ is z-superstable for M. Similarly $\vec{c}$ is critical if and only if $\lfloor ML^{-1}\vec{c} \rceil$ is critical for M.*

Theorem 5. ***[Bounding the criticals]*** *Suppose G_ϕ is a signed graph where each non-sink vertex is adjacent to q, has degree m, and is incident to m' negative edges. Then, $(m(2m'+1) - 1)\vec{1}$ is the maximum critical configuration.*

Theorem 6. ***[Relating χ and z-superstables]*** *Suppose G_ϕ is any connected signed graph. Then the set of χ-superstable configurations is the same as the set of z-superstable configurations.*

Proposition 1. ***[Computing the Identity]*** *Let $G = G_\phi$ be a signed graph with underlying unsigned graph $|G|$. Then the identity of the critical group is given by*

$$I_{\mathcal{K}(G)} = LM^{-1} I_{\mathcal{K}(|G|)}.$$

Further Questions

- Can we efficiently calculate the z-superstable configurations of a signed graph?
- Can we find a bijection between the z-superstables and critical configurations of a graph, similar to that for unsigned graphs as in [2]?
- How does *vertex switching* on a signed graph affect critical configurations?

References

1. Guzmán, J. & Klivans, C. Chip firing on general invertible matrices. *SIAM J. Discrete Math.* **30**, 1115–1127 (2016).
2. Klivans, C. J. *The mathematics of chip-firing.* xii+295 (CRC Press, Boca Raton, FL, 2019).

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